Towards Minimal Models of Elliptic Fourfolds

David Wen
University of California, Santa Barbara
Advisor: David R. Morrison

Background

Theorem

[3, Thm 1.1] Let $X_0 \to S_0$ be an elliptic threefold which is not uniruled. Then there exists a birationally equivalent fibration $\bar{\pi} : \bar{X} \to \bar{S}$, such that $\bar{X}$ has at worst terminal and $\bar{S}$ log terminal singlarities. Futhermore $K_{\bar{X}}$ is nef and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{S}} + \bar{\Lambda})$, where $\bar{\Lambda}$ is a $\mathbb{Q}$-boundary divisor. Thus the canonical bundle is a pullback of a $\mathbb{Q}$-bundle on $\bar{S}$.

The proof of the above theorem is best followed with a commutative diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "\mu"] \arrow[d, "\pi"'] \arrow[dr, "\epsilon"] & \bar{X} \arrow[d, "\bar{\pi}"] \\ (B,\Delta) \arrow[r, dashed, "\psi"'] & (\bar{B},\bar{\Delta}) \end{tikzcd}$$

Grassi began with the following canonical bundle formula for elliptic fibrations below from [2]

$$K_X = \pi^*\left(K_B + \pi_*(K_{X/B}) + \Sigma_i \left(\frac{m_i - 1}{m_i} Y_i\right)\right) + E - G \quad (1)$$

After possibly some blow ups, we have $\pi_*(K_{X/B})$ is a sum of divisors supporting the singular fibers of the elliptic fibrations with coefficients in $[0,1) \cap \mathbb{Q}$ determined by the Kodaira type. Letting $\Lambda = \pi_*(K_{X/B}) + \Sigma_i \left(\frac{m_i-1}{m_i} Y_i\right)$, we have that (B,Λ) is a log terminal pair.
After some work, formula 1 leads to a Fujita-Zariski decomposition of K_X. By running the log minimal model program for surfaces on (B,Λ) to get $\psi : B \to \bar{B}$, and then the relative minimal model program on $\epsilon : X \to \bar{B}$, this gives a relative minimal model $\bar{\pi} : \bar{X} \to \bar{B}$. By analyzing the decomposition of K_X and how these birational maps affect it, this results in the formula $K_{\bar{X}} = \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$. This shows that $\bar{X}$ is a minimal model of X and is an elliptic fibration over $\bar{B}$.

Towards a Higher Dimensional Generalization

One main obstruction generalizing to higher dimensional elliptic fibrations, is the fact that, in higher dimensions, divisors may not have a generalized Zariski decomposition. Above, the Zariski decomposition for surfaces was used toward showing that formula 1 results in a Zariski decomposition for the canonical divisor of the elliptic threefold. Fortunately, there is a relation between minimal models and Zariski decomposition with the following theorem.

Theorem

[1, Thm 1.5] Assume the log minimal model program for $\mathbb{Q}$-factorial divisorial log terminal pairs in dimension $n-1$. Let (X,Δ) be log canonical pair of dimension n then $K_X + \Delta$ birationally has a Fujita-Zariski Decomposition if and only if (X,Δ) has a log minimal model.

If (B,Δ) has a log minimal model (for example in the log terminal threefold case), then there is a sequence of blow ups $g : \tilde{B} \to B$ such that $g^*(K_B + \Delta)$ has a Fujita-Zariski decomposition. These blow ups will change the fibration and might result in more singular fibers which might affect the canonical bundle formula. For the threefold case, analysis of this was handled via Weierstrass models in [4] and we use a similar analysis for the fourfold case. This let's us verify that birationally the canonical divisor of a Weierstrass model of dimension 4 has a Fujita-Zariski decomposition and we can explicitly write it.

Commutative Diagram towards Generalization

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\tilde{g}"'] \arrow[dr, dashed, "\mu"] \arrow[dd, "\tilde{\pi}"] \arrow[dddr, "\epsilon"] & \\ X \arrow[dd, "\pi"'] & & \bar{X} \arrow[dd, "\bar{\pi}"] \\ & (\tilde{B},\tilde{\Delta}) \arrow[dl, "g"'] \arrow[dr, "h"] & \\ (B,\Delta) \arrow[rr, dashed, "\psi"] & & (\bar{B},\bar{\Delta}) \end{tikzcd}$$

We start with X being a Weierstrass model over B and $\Delta = \pi_* K_{X/B} = \Sigma_i a_i D_i$ having simple normal crossing, where D_i supports the singular elliptic fibers of π from the Kodaira classification and $a_i \in \mathbb{Q} \cap [0,1)$ are determined by the Kodaira type.

$(\bar{B},\bar{\Delta})$ is a log minimal model of (B,Δ) from running log MMP with ψ the sequence of log contractions and log flips.

$\tilde{B}$ is a common log resolution from the proof of [1, Theorem 1.5], so that $g^*(K_B + \Delta)$ has a Fujita-Zariski Decomposition.

$\tilde{X}$ is obtained by taking the fiber product of X and $\tilde{B}$ and then resolving the singularities. So $\tilde{\pi}$ is a elliptic fibration between smooth projective varieties.

$\bar{X}$ is obtained from running the minimal model program on $\tilde{X}$. By analyzing divisors and using the Fujita-Zariski decomposition, we get a partial generalization of Grassi's theorem.

Results

Theorem

(DW) Let $\pi : X \to B$ be a Weierstrass model, $\Delta = \pi_* K_{X/B}$ where (B,Δ) is a log terminal threefold with a log minimal model $(\bar{B},\bar{\Delta})$. Then there exists a birationally equivalent rational fibration $\bar{\pi} : \bar{X} \dashrightarrow \bar{B}$, such that $\bar{X}$ has at worst terminal singularities and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.

Theorem

(DW) With the assumptions from the above theorem, the canonical model of $\bar{X}$ is isomorphic to the log canonical model of $(\bar{B},\bar{\Delta})$. Equivalently, the canonical ring of $\bar{X}$ is isomorphic to the log canonical ring of $(\bar{B},\bar{\Delta})$.

Future Direction

My next goal is to generalize the theorem to elliptic fourfolds with no rational sections (this means it will not be birational to a Weierstrass Model) and also to understand the behavior of the elliptic fourfolds with respect birational maps on the base. One interesting question would be: What are the conditions to obtain a minimal model of an elliptic fourfold that is also equidimensional over the base? Do these results generalize to high dimensional fibers and varieties?

References

[1] Caucher Birkar.
On existence of log minimal models and weak Zariski decompositions.
Math. Ann., 354(2):787–799, 2012.

[2] Takao Fujita.
Zariski decomposition and canonical rings of elliptic threefolds.
J. Math. Soc. Japan, 38(1):19–37, 1986.

[3] A. Grassi.
On minimal models of elliptic threefolds.
Math. Ann., 290(2):287–301, 1991.

[4] Rick Miranda.
Smooth models for elliptic threefolds.
In *The birational geometry of degenerations (Cambridge, Mass., 1981)*, volume 29 of *Progr. Math.*, pages 85–133. Birkhäuser, Boston, Mass., 1983.